

Time-Periodic Solutions of the Einstein's Field Equations II

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In this paper, we construct several kinds of new time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. The singularities of these new time-periodic solutions are investigated and some new physical phenomena are found. The applications of these solutions in modern cosmology and general relativity can be expected.

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1. Introduction. The Einstein's field equations are the fundamental equations in general relativity and play an essential role in cosmology. This paper concerns the time-periodic solutions of the following vacuum Einstein's field equations

$$G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad (1)$$

or equivalently,

$$R_{\mu\nu} = 0, \quad (2)$$

where $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and $G_{\mu\nu}$ is the Einstein tensor.

It is well known that the exact solutions of the Einstein's field equations play a crucial role in general relativity and cosmology. Typical examples are the Schwarzschild solution and Kerr solution. Although many interesting and important solutions have been obtained (see, e.g., [1] and [5]), there are still many fundamental open problems. One such problem is *if there exists a "time-periodic" solution, which contains physical singularities such as black hole, to the Einstein's field equations.* This paper continues the discussion of this problem.

The first time-periodic solution of the vacuum Einstein's field equations was constructed by the first two authors in [3]. The solution presented in [3] is time-periodic, and describes a regular space-time, which has

vanishing Riemann curvature tensor but is inhomogeneous, anisotropic and not asymptotically flat. In particular, this space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to steady event horizons, time-periodic event horizon and has some interesting new physical phenomena.

In this paper, we focus on finding the time-periodic solutions, which contain physical singularities such as black hole to the vacuum Einstein's field equations (1). We shall construct three kinds of new time-periodic solutions of the vacuum Einstein's field equations (1) whose Riemann curvature tensors vanish, keep finite or go to the infinity at some points in these space-times respectively. The singularities of these new time-periodic solutions are investigated and new physical phenomena are found. Moreover, the applications of these solutions in modern cosmology and general relativity may be expected. In the forthcoming paper [4], we shall construct a time-periodic solution of the Einstein's field equations with black hole, which describes the time-periodic cosmology with many new and interesting physical phenomena.

2. Procedure of finding new solutions.

We consider the metric of the following form

$$(g_{\mu\nu}) = \begin{pmatrix} u & v & p & 0 \\ v & 0 & 0 & 0 \\ p & 0 & f & 0 \\ 0 & 0 & 0 & h \end{pmatrix}, \quad (3)$$

where u, v, p, f and h are smooth functions of the coordinates (t, x, y, z) . It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \triangleq \det(g_{\mu\nu}) = -v^2 fh. \quad (4)$$

Throughout this paper, we assume that

$$g < 0. \quad (H)$$

Without loss of generality, we may suppose that f and g keep the same sign, for example,

$$f < 0 \text{ (resp. } f > 0) \text{ and } h < 0 \text{ (resp. } g > 0). \quad (5)$$

In what follows, we solve the Einstein's field equations (2) under the framework of the Lorentzian metric of the form (3).

By a direct calculation, we have the Ricci tensor

$$R_{11} = -\frac{1}{2} \left\{ \frac{v_x}{v} \left(\frac{f_x}{f} + \frac{h_x}{h} \right) + \frac{1}{2} \left[\left(\frac{f_x}{f} \right)^2 + \left(\frac{h_x}{h} \right)^2 \right] - \left(\frac{f_{xx}}{f} + \frac{h_{xx}}{h} \right) \right\}. \quad (6)$$

It follows from (2) that

$$\frac{v_x}{v} \left(\frac{f_x}{f} + \frac{h_x}{h} \right) + \frac{1}{2} \left[\left(\frac{f_x}{f} \right)^2 + \left(\frac{h_x}{h} \right)^2 \right] - \left(\frac{f_{xx}}{f} + \frac{h_{xx}}{h} \right) = 0. \quad (7)$$

This is an ordinary differential equation of first order on the unknown function v . Solving (7) gives

$$v = V(t, y, z) \exp \left\{ \int \Theta(t, x, y, z) dx \right\}, \quad (8)$$

where

$$\Theta = \left[\frac{f_{xx}}{f} + \frac{h_{xx}}{h} - \frac{1}{2} \left(\frac{f_x}{f} \right)^2 - \frac{1}{2} \left(\frac{h_x}{h} \right)^2 \right] \frac{fh}{(fh)_x},$$

and $V = V(t, y, z)$ is an integral function depending on t, y and z . Here we assume that

$$(fh)_x \neq 0. \quad (9)$$

In particular, taking the ansatz

$$f = -K(t, x)^2, \quad h = N(t, y, z)K(t, x)^2 \quad (10)$$

and substituting it into (8) yields

$$v = VK_x. \quad (11)$$

By the assumptions (H) and (9), we have

$$V \neq 0, \quad K \neq 0, \quad K_x \neq 0. \quad (12)$$

Noting (10) and (11), by a direct calculation we obtain

$$R_{13} = -\frac{V_z K_x}{KV}. \quad (13)$$

It follows from (2) that

$$R_{13} = 0.$$

Combining (12) and (13) gives

$$V_z = 0. \quad (14)$$

This implies that the function V depends only on t, y but is independent of x and z . Noting (10)-(11) and using (14), we calculate

$$R_{12} = -\frac{1}{2V} \left(\frac{p_{xx}}{K_x} - \frac{K_{xx}p_x}{K^2_x} - \frac{2pK_x}{K^2} + \frac{2K_x V_y}{K} \right). \quad (15)$$

Solving p from the equation $R_{12} = 0$ yields

$$p = AK^2 + V_y K + \frac{B}{K}, \quad (16)$$

where A and B are integral functions depending on t, y and z . Noting (10)-(11) and using (14) and (16), we observe that the equation $R_{23} = 0$ is equivalent to

$$B_z - 2K^3 A_z = 0. \quad (17)$$

Since K is a function depending only on t, x , and A, B are functions depending on t, y and z , we can obtain that

$$B = 2K^3 A + C(t, x, y), \quad (18)$$

where C is an integral function depending on t, x and y . For simplicity, we take

$$A = B = C = 0. \quad (19)$$

Thus, (16) simplifies to

$$p = V_y K. \quad (20)$$

From now on, we assume that the function N only depends on y , that is to say,

$$N = N(y). \quad (21)$$

Substituting (10)-(11), (14) and (20)-(21) into the equation $R_{02} = 0$ yields

$$u_x V_y + V(u_{yx} - 4V_y K_{xt}) = 0. \quad (22)$$

Solving u from the equation (22) leads to

$$u = 2K_t V. \quad (23)$$

Noting (10)-(11), (14), (20)-(21) and (23), by a direct calculation we obtain

$$R_{03} = 0, \quad (24)$$

$$\begin{cases} R_{22} = (4N^2 V^2)^{-1} [2NV^2 N_{yy} - 4N^2 V V_{yy} \\ \quad + 4N^2 V_y^2 - 2NV N_y V_y - V^2 N_y^2], \\ R_{33} = -(4NV^2)^{-1} [2NV^2 N_{yy} - 4N^2 V V_{yy} \\ \quad + 4N^2 V_y^2 - 2NV N_y V_y - V^2 N_y^2] \end{cases} \quad (25)$$

and

$$R_{00} = (2KNV^2)^{-1} [4NV_t V_y^2 + 2NV^2 V_{tyy} - 2NV V_t V_{yy} \\ - 4NV V_y V_{ty} - V N_y V_t V_y + V^2 N_y V_{ty}]. \quad (26)$$

Therefore, under the assumptions mentioned above, the Einstein's field equations (2) are reduced to

$$-\frac{N_{yy}}{N} + \frac{1}{2} \left(\frac{N_y}{N} \right)^2 + 2 \frac{V_{yy}}{V} + \frac{N_y V_y}{NV} - 2 \left(\frac{V_y}{V} \right)^2 = 0 \quad (27)$$

and

$$4V_y^2 V_t + 2V^2 V_{yyt} - 2V V_{yy} V_t - 4V V_y V_{yt} \\ - \frac{V V_y V_t N_y}{N} + \frac{V^2 V_{yt} N_y}{N} = 0. \quad (28)$$

On the other hand, (27) can be rewritten as

$$2 \left(\frac{V_y}{V} \right)_y + \frac{V_y N_y}{VN} - \left(\frac{N_y}{N} \right)_y - \frac{1}{2} \left(\frac{N_y}{N} \right)^2 = 0 \quad (29)$$

and (28) is equivalent to

$$2 \left(\frac{V_y}{V} \right)_{yt} + \left(\frac{V_y}{V} \right)_t \frac{N_y}{N} = 0. \quad (30)$$

Noting (21) and differentiating (29) with respect to t gives (30) directly. This shows that (29) implies (30). Hence in the present situation, the Einstein's field equations (2) are essentially (29). Solving V from the equation (29) yields

$$V = w(t) |N(y)|^{1/2} \exp \left\{ q(t) \int |N(y)|^{-1/2} dy \right\}, \quad (31)$$

where $w = w(t)$ and $q = q(t)$ are two integral functions only depending on t . Thus, we can obtain the following solution of the vacuum Einstein's field equations in the coordinates (t, x, y, z)

$$ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \quad (32)$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 2K_t V & K_x V & K V_y & 0 \\ K_x V & 0 & 0 & 0 \\ K V_y & 0 & -K^2 & 0 \\ 0 & 0 & 0 & N K^2 \end{pmatrix}, \quad (33)$$

in which $N = N(y)$ is an arbitrary function of y , $K = K(t, x)$ is an arbitrary function of t, x , and V is given by (31).

By calculations, the Riemann curvature tensor reads

$$R_{\alpha\beta\mu\nu} = 0, \quad \forall \alpha\beta\mu\nu \neq 0202 \text{ or } 0303, \quad (34)$$

while

$$R_{0202} = K w q q' |N|^{-1/2} \exp \left\{ q \int |N|^{-1/2} dy \right\} \quad (35)$$

and

$$R_{0303} = K w q q' |N|^{1/2} \exp \left\{ q \int |N|^{-1/2} dy \right\}. \quad (36)$$

3. Time-periodic solutions. This section is devoted to constructing some new time-periodic solutions of the vacuum Einstein's field equations.

3.1 Regular time-periodic space-times with vanishing Riemann curvature tensor. Take $q = \text{constant}$ and let $V = \rho(t)\kappa(y)$, where κ is defined by

$$\kappa(y) = c_1 \sqrt{|N|} \exp \left\{ c_2 \int |N|^{-1/2} dy \right\}, \quad (37)$$

in which c_1 and c_2 are two integrable constants. In this case, the solution to the vacuum Einstein's filed equations in the coordinates (t, x, y, z) reads

$$ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \quad (38)$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 2\rho\kappa\partial_t K & \rho\kappa\partial_x K & \rho K\partial_y \kappa & 0 \\ \rho\kappa\partial_x K & 0 & 0 & 0 \\ \rho K\partial_y \kappa & 0 & -K^2 & 0 \\ 0 & 0 & 0 & NK^2 \end{pmatrix}. \quad (39)$$

Theorem 1 *The vacuum Einstein's filed equations (2) have a solution described by (38) and (39), and the Riemann curvature tensor of this solution vanishes. ■*

As an example, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = 0, \\ K(t, x) = e^x \sin t, \\ N(y) = -(2 + \sin y)^2. \end{cases} \quad (40)$$

In the present situation, we obtain the following solution of the vacuum Einstein's filed equations (2)

$$(\eta_{\mu\nu}) = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & 0 \\ \eta_{01} & 0 & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} & 0 \\ 0 & 0 & 0 & \eta_{33} \end{pmatrix}, \quad (41)$$

where

$$\begin{cases} \eta_{00} = 2e^x(2 + \sin y) \cos^2 t, \\ \eta_{01} = \frac{1}{2}e^x(2 + \sin y) \sin(2t), \\ \eta_{02} = \frac{1}{2}e^x \cos y \sin(2t), \\ \eta_{22} = -[e^x \sin t]^2, \\ \eta_{33} = -[e^x(2 + \sin y) \sin t]^2. \end{cases} \quad (42)$$

By (4),

$$\eta \triangleq \det(\eta_{\mu\nu}) = -\frac{1}{4}e^{6x}(2 + \sin y)^4 \sin^4 t \sin^2(2t). \quad (43)$$

Property 1 The solution (41) of the vacuum Einstein's filed equations (2) is time-periodic. ■

Proof. In fact, the first equality in (42) implies that

$$\eta_{00} > 0 \quad \text{for } t \neq k\pi + \pi/2 \quad (k \in \mathbb{N}) \quad \text{and } x \neq -\infty.$$

On the other hand, by direct calculations,

$$\begin{vmatrix} \eta_{00} & \eta_{01} \\ \eta_{01} & 0 \end{vmatrix} = -\frac{1}{4}e^{2x}(2 + \sin y)^2 \sin^2(2t) < 0,$$

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} \\ \eta_{01} & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} \end{vmatrix} = -\eta_{01}^2 \eta_{22} > 0$$

and

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} & 0 \\ \eta_{01} & 0 & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} & 0 \\ 0 & 0 & 0 & \eta_{33} \end{vmatrix} = -\eta_{01}^2 \eta_{22} \eta_{33} < 0$$

for $t \neq k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x \neq -\infty$.

In Property 3 below, we will show that $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) are the singularities of the space-time described by (41), but they are not essential (or say, physical) singularities, these non-essential singularities correspond to the event horizons of the space-time described by (41) with (42); while, when $x = -\infty$, the space-time (41) degenerates to a point.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (42) that the Lorentzian metric

$$ds^2 = (dt, dx, dy, dz)(\eta_{\mu\nu})(dt, dx, dy, dz)^T \quad (44)$$

is indeed a time-periodic solution of the vacuum Einstein's field equations (2), where $(\eta_{\mu\nu})$ is given by (41). This proves Property 1. ■

Noting (34)-(36) and the second equality in (40) gives **Property 2** The Lorentzian metric (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)) describes a regular space-time, this space-time is Riemannian flat, that is to say, its Riemann curvature tensor vanishes. ■

Remark 1 *The first time-periodic solution to the Einstein's field equations was constructed by Kong and Liu [3]. The time-periodic solution presented in [3] also has the vanishing Riemann curvature tensor.*

It follows from (43) that the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x = \pm\infty$ are singularities of the

space-time (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)), however, by Property 2, these singularities are not physical (or say, not essential). According to the definition of event horizon (see e.g., Wald [6]), it is easy to show that the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x = +\infty$ are the event horizons of the space-time (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)). Therefore, we have

Property 3 The Lorentzian metric (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)) does not contain any essential singularity. These non-essential singularities consist of the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x = \pm\infty$. The singularities $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x = +\infty$ correspond to the event horizons, while, when $x = -\infty$, the space-time (44) degenerates to a point. ■

We now investigate the physical behavior of the space-time (44).

Fixing y and z , we get the induced metric

$$ds^2 = \eta_{00}dt^2 + 2\eta_{01}dtdx. \quad (45)$$

Consider the null curves in the (t, x) -plan, which are defined by

$$\eta_{00}dt^2 + 2\eta_{01}dtdx = 0. \quad (46)$$

Noting (42) gives

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\tan t. \quad (47)$$

Thus, the null curves and light-cones are shown in Figure 1.

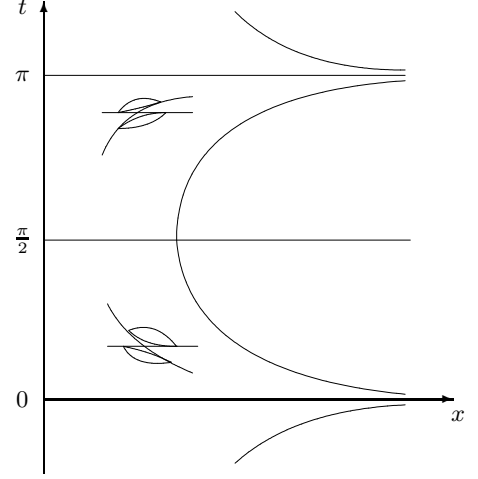


FIG. 1: Null curves and light-cones in the domains $0 < t < \pi/2$ and $\pi/2 < t < \pi$.

We next study the geometric behavior of the t -slices.

For any fixed $t \in \mathbb{R}$, it follows from (44) that the induced metric of the t -slice reads

$$\begin{aligned} ds^2 &= \eta_{22}dy^2 + \eta_{33}dz^2 \\ &= -e^{2x} \sin^2 t [dy^2 + (2 + \sin y)^2 dz^2]. \end{aligned} \quad (48)$$

When $t = k\pi$ ($k \in \mathbb{N}$), the metric (48) becomes

$$ds^2 = 0.$$

This implies that the t -slice reduces to a point. On the other hand, in the present situation, the metric (44) becomes

$$ds^2 = 2e^x(2 + \sin y)dt^2.$$

When $t \neq k\pi$ ($k \in \mathbb{N}$), (48) shows that the t -slice is a three-dimensional cone-like manifold centered at $x = -\infty$.

3.2 Regular time-periodic space-times with non-vanishing Riemann curvature tensor. We next construct the regular time-periodic space-times with non-vanishing Riemann curvature tensor.

To do so, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = \sin t, \\ K(x, t) = e^x \sin t, \\ N = -\frac{1}{(2 + \sin y)^2}. \end{cases} \quad (49)$$

Then, by (31),

$$V = \frac{\cos t \exp \{(2y - \cos y) \sin t\}}{2 + \sin y}.$$

Thus, in the present situation, we have the following solution of the vacuum Einstein's field equations (2)

$$\tilde{\eta}_{\mu\nu} = \begin{pmatrix} \tilde{\eta}_{00} & \tilde{\eta}_{01} & \tilde{\eta}_{02} & 0 \\ \tilde{\eta}_{01} & 0 & 0 & 0 \\ \tilde{\eta}_{02} & 0 & \tilde{\eta}_{22} & 0 \\ 0 & 0 & 0 & \tilde{\eta}_{33} \end{pmatrix}, \quad (50)$$

where

$$\begin{cases} \tilde{\eta}_{00} = \frac{2e^x \cos^2 t \exp \{(2y - \cos y) \sin t\}}{2 + \sin y}, \\ \tilde{\eta}_{01} = \frac{e^x \sin(2t) \exp \{(2y - \cos y) \sin t\}}{2(2 + \sin y)}, \\ \tilde{\eta}_{02} = e^x \left\{ \sin t \cos t - \frac{\cos t \cos y}{(2 + \sin y)^2} \right\} \sin t \\ \quad \times \exp \{(2y - \cos y) \sin t\}, \\ \tilde{\eta}_{22} = -e^{2x} \sin^2 t, \\ \tilde{\eta}_{33} = -\frac{e^{2x} \sin^2 t}{(2 + \sin y)^2}. \end{cases} \quad (51)$$

By (4),

$$\begin{aligned} \tilde{\eta} &\triangleq \det(\tilde{\eta}_{\mu\nu}) = -(\tilde{\eta}_{01})^2 \tilde{\eta}_{22} \tilde{\eta}_{33} \\ &= -\frac{e^{6x+2(2y-\cos y)\sin t} \sin^2(2t) \sin^4 t}{4(2 + \sin y)^4}. \end{aligned} \quad (52)$$

Introduce

$$\Delta(t, x, y) = 6x + 2(2y - \cos y) \sin t.$$

Thus, it follows from (52) that

$$\tilde{\eta} < 0 \quad (53)$$

for $t \neq k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$) and $\Delta \neq -\infty$. It is obvious that the hypersurfaces $t = k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$) and $\Delta = \pm\infty$ are the singularities of the space-time described

by (50) with (51). As in Subsection 3.1, we can prove that the hypersurfaces $t = k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$) are not essential (or say, physical) singularities, these non-essential singularities correspond to the event horizons of the space-time described by (50) with (51).

Similar to Property 1, we have

Property 4 The solution (50) (in which $(\tilde{\eta}_{\mu\nu})$ is given by (51)) of the vacuum Einstein's field equations (2) is time-periodic. ■

Similar to Property 2, we have

Property 5 The Lorentzian metric (50) (in which $(\tilde{\eta}_{\mu\nu})$ is given by (51)) describes a regular space-time, this space-time has a non-vanishing Riemann curvature tensor. ■

Proof. In the present situation, by (34)

$$R_{\alpha\beta\mu\nu} = 0, \quad \forall \alpha\beta\mu\nu \neq 0202 \text{ or } 0303, \quad (54)$$

while

$$\begin{aligned} R_{0202} &= e^x (2 + \sin y) \cos^2 t \sin^2 t \\ &\quad \times \exp \{(2y - \cos y) \sin t\}, \end{aligned} \quad (55)$$

and

$$R_{0303} = \frac{e^x \cos^2 t \sin^2 t \exp \{(2y - \cos y) \sin t\}}{2 + \sin y}. \quad (56)$$

Property 5 follows from (54)-(56) directly. Thus the proof is completed. □

In particular, when $t \neq k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$), it follows from (55) and (56) that

$$R_{0202}, R_{0303} \longrightarrow \infty \quad \text{as } x + (2y - \cos y) \sin t \rightarrow \infty. \quad (57)$$

However, a direct calculation gives

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \equiv 0. \quad (58)$$

Thus, we obtain

Property 6 The Lorentzian metric (50) (in which $(\tilde{\eta}_{\mu\nu})$ is given by (51)) does not contain any essential singularity. These non-essential singularities consist of the hypersurfaces $t = k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$) and $\Delta = \pm\infty$, in

which the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) are the event horizons. Moreover, the Riemann curvature tensor satisfies the properties (57) and (58). ■

We next analyze the singularity behavior of $\Delta = \pm\infty$.

Case 1: Fixing $y \in \mathbb{R}$, we observe that

$$\Delta \rightarrow \pm\infty \iff x \rightarrow \pm\infty.$$

This situation is similar to the case $x \rightarrow \pm\infty$ discussed in Subsection 3.1. That is to say, $x = +\infty$ corresponds to the event horizon, while, when $x \rightarrow -\infty$, the space-time (50) with (51) degenerates to a point.

Case 2: Fixing $x \in \mathbb{R}$, we observe that

$$\Delta \rightarrow \pm\infty \iff y \rightarrow \pm\infty.$$

In the present situation, it holds that

$$t \neq k\pi \quad (k \in \mathbb{N}).$$

Without loss of generality, we may assume that

$$\sin t > 0.$$

For the case that $\sin t < 0$, we have a similar discussion.

Thus, noting (57), we have

$$R_{0202}, R_{0303} \longrightarrow \infty \quad \text{as } y \rightarrow \infty.$$

Moreover, by the definition of the event horizon we can show that $y = +\infty$ is not a event horizon. On the other hand, when $y \rightarrow -\infty$, the space-time (50) with (51) degenerates to a point.

Case 3: For the situation that $x \rightarrow \pm\infty$ and $y \rightarrow \pm\infty$ simultaneously, we have a similar discussion, here we omit the details.

For the space-time (50) with (51), the null curves and light-cones are shown just as in Figure 1. On the other hand, for any fixed $t \in \mathbb{R}$, the induced metric of the t -slice reads

$$\begin{aligned} ds^2 &= \tilde{\eta}_{22} dy^2 + \tilde{\eta}_{33} dz^2 \\ &= -e^{2x} \sin^2 t [dy^2 + (2 + \sin y)^{-2} dz^2]. \end{aligned} \quad (59)$$

Obviously, in the present situation, the t -slice possesses similar properties shown in the last paragraph in Subsection 3.1.

In particular, if we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) with $t \in \mathbb{R}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\varphi \in [-\pi/2, \pi/2]$, then the metric (50) with (51) describes a regular time-periodic space-time with non-vanishing Riemann curvature tensor. This space-time does not contain any essential singularity, these non-essential singularities consist of the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) which are the event horizons. The Riemann curvature tensor satisfies (58) and

$$R_{0202}, R_{0303} \longrightarrow \infty \quad \text{as } r \rightarrow \infty.$$

Moreover, when $t \neq k\pi$ ($k \in \mathbb{N}$), the t -slice is a three dimensional bugle-like manifold with the base at $x = 0$; while, when $t = k\pi$ ($k \in \mathbb{N}$), the t -slice reduces to a point.

3.3 Time-periodic space-times with physical singularities. This subsection is devoted to constructing the time-periodic space-times with physical singularities.

To do so, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = \sin t, \\ K(x, t) = \frac{\sin t}{x^2}, \\ N = -\frac{1}{(2 + \cos y)^2}. \end{cases} \quad (60)$$

Then, by (31) we have

$$V = \frac{\cos t \exp \{(2y + \sin y) \sin t\}}{2 + \cos y}.$$

Thus, in the present situation, the solution of the vacuum Einstein's field equations (2) in the coordinates (t, x, y, z) reads

$$ds^2 = (dt, dx, dy, dz)(\hat{\eta}_{\mu\nu})(dt, dx, dy, dz)^T, \quad (61)$$

where

$$(\hat{\eta}_{\mu\nu}) = \begin{pmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} & 0 \\ \hat{\eta}_{01} & 0 & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} & 0 \\ 0 & 0 & 0 & \hat{\eta}_{33} \end{pmatrix}, \quad (62)$$

in which

$$\begin{cases} \hat{\eta}_{00} = \frac{2 \cos^2 t \exp \{(\sin y + 2y) \sin t\}}{(2 + \cos y)x^2}, \\ \hat{\eta}_{01} = -\frac{\sin(2t) \exp \{(\sin y + 2y) \sin t\}}{(2 + \cos y)x^3}, \\ \hat{\eta}_{02} = \frac{\sin t}{x^2} \left\{ \frac{\cos t \sin y}{(2 + \cos y)^2} + \frac{\sin(2t)}{2} \right\} \times \\ \quad \exp \{(\sin y + 2y) \sin t\}, \\ \hat{\eta}_{22} = -\frac{\sin^2 t}{x^4}, \\ \hat{\eta}_{33} = -\frac{\sin^2 t}{(2 + \cos y)^2 x^4}. \end{cases} \quad (63)$$

By (4), we have

$$\begin{aligned} \hat{\eta} &\triangleq \det(\hat{\eta}_{\mu\nu}) = -(\hat{\eta}_{01})^2 \hat{\eta}_{22} \hat{\eta}_{33} \\ &= -\frac{e^{2(2y + \sin y) \sin t} \sin^2(2t) \sin^4 t}{x^{14} (2 + \cos y)^4}. \end{aligned} \quad (64)$$

It follows from (63) that

$$\hat{\eta} < 0 \quad (65)$$

for $t \neq k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x \neq 0$. Obviously, the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x = 0$ are the singularities of the space-time described by (61) with (62)-(63). As before, we can prove that the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) are not essential (or, say, physical) singularities, and these non-essential singularities correspond to the event horizons of the space-time described by (61) with (62)-(63), however $x = 0$ is an essential (or, say, physical) singularity (see Property 8 below).

Similar to Property 1, we have

Property 7 The solution (61) (in which $(\hat{\eta}_{\mu\nu})$ is given by (62) and (63)) of the vacuum Einstein's field equations (2) is time-periodic. ■

Proof. In fact, the first equality in (63) implies that

$$\hat{\eta}_{00} > 0 \quad \text{for } t \neq k\pi + \pi/2 \quad (k \in \mathbb{N}) \text{ and } x \neq 0. \quad (66)$$

On the other hand, by direct calculations we have

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} \\ \hat{\eta}_{01} & 0 \end{vmatrix} = -\hat{\eta}_{01}^2 < 0, \quad (67)$$

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} \\ \hat{\eta}_{01} & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} \end{vmatrix} = -\hat{\eta}_{01}^2 \hat{\eta}_{22} > 0 \quad (68)$$

and

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} & 0 \\ \hat{\eta}_{01} & 0 & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} & 0 \\ 0 & 0 & 0 & \hat{\eta}_{33} \end{vmatrix} = -\hat{\eta}_{01}^2 \hat{\eta}_{22} \hat{\eta}_{33} < 0 \quad (69)$$

for $t \neq k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) and $x \neq 0$.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (63) that the Lorentzian metric (61) is indeed a time-periodic solution of the vacuum Einstein's field equations (2), where $(\hat{\eta}_{\mu\nu})$ is given by (63). This proves Property 7. □

Property 8 When $t \neq k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$), for any fixed $y \in \mathbb{R}$ it holds that

$$R_{0202} \rightarrow +\infty \quad \text{and} \quad R_{0303} \rightarrow +\infty, \quad \text{as } x \rightarrow 0. \quad (70)$$

■

Proof. By direct calculations, we obtain from (35) and (36) that

$$R_{0202} = \frac{(2 + \cos y) \sin^2(2t) \exp \{(\sin y + 2y) \sin t\}}{4x^2}, \quad (71)$$

and

$$R_{0303} = \frac{\sin^2(2t) \exp \{(\sin(y) + 2y) \sin t\}}{4x^2(2 + \cos y)}. \quad (72)$$

(70) follows from (71) and (72) directly. The proof is finished. □

On the other hand, a direct calculation yields

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \equiv 0. \quad (73)$$

Therefore, we have

Property 9 The Lorentzian metric (61) describes a time-periodic space-time, this space-time contains two kinds of singularities: the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$), which are non-essential singularities and correspond to the event horizons, and $x = 0$, which is an essential (or, say, physical) singularity. ■

We now analyze the behavior of the singularities of the space-time characterized by (61) with (63).

By (64), we shall investigate the following cases: (a) $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$); (b) $y \rightarrow \pm\infty$; (c) $x \rightarrow \pm\infty$; (d) $x \rightarrow 0$.

Case a: $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$). According to the definition of the event horizon, the hypersurfaces $t = k\pi$, $k\pi + \pi/2$ ($k \in \mathbb{N}$) are the event horizons of the space-time described by (61) with (63).

Case b: $y \rightarrow \pm\infty$. Noting (64), in this case we may assume that $t \neq k\pi$ ($k \in \mathbb{N}$) (if $t = k\pi$, then the situation becomes trivial). Without loss of generality, we may assume that $\sin t > 0$. Therefore, it follows from (71) and (72) that, for any fixed $x \neq 0$ it holds that

$$R_{0202}, R_{0303} \longrightarrow \infty \quad \text{as } y \rightarrow +\infty \quad (74)$$

and

$$R_{0202}, R_{0303} \longrightarrow 0 \quad \text{as } y \rightarrow -\infty. \quad (75)$$

(74) implies that $y = +\infty$ is also a essential singularity, while $y = -\infty$ is not because of (75).

Case c: $x \rightarrow \pm\infty$. By (63), in this case the space-time characterized by (61) reduces to a point.

Case d: $x \rightarrow 0$. Property 8 shows that $x = 0$ is a physical singularity. This is the biggest difference between the space-times presented in Subsections 3.1-3.2 and the one given this subsection. In order to illustrate its physical meaning, we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) with $t \in \mathbb{R}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\varphi \in [-\pi/2, \pi/2]$. In the coordinates (t, r, θ, φ) , the metric (61) with (63) describe a time-periodic space-time which possesses three kind of singularities:

- (i) $t \neq k\pi$ ($k \in \mathbb{N}$): they are the event horizons;
- (ii) $r \rightarrow +\infty$: the space-time degenerates to a point;
- (iii) $r \rightarrow 0$: it is a physical singularity.

For the case (iii), in fact Property 8 shows that every point in the set

$$\mathfrak{S}_B \stackrel{\Delta}{=} \{(t, r, \theta, \varphi) \mid r = 0, t \neq k\pi, k\pi + \pi/2 (k \in \mathbb{N})\}$$

is a singular point. Noting (34) and (70), we name the set of singular points \mathfrak{S}_B as a *quasi-black-hole*. Property 8 also shows that the space-time (61) is not homogenous and not asymptotically flat. This space-time perhaps has some new applications in cosmology due to the recent WMAP data, since the recent WMAP data show that our Universe exists anisotropy (see [2]). This inhomogenous property of the new space-time (61) may provide a way to give an explanation of this phenomena.

We next investigate the physical behavior of the space-time (61).

Fixing y and z , we get the induced metric

$$ds^2 = \hat{\eta}_{00} dt^2 + 2\hat{\eta}_{01} dt dx. \quad (76)$$

Consider the null curves in the (t, x) -plan defined by

$$\hat{\eta}_{00} dt^2 + 2\hat{\eta}_{01} dt dx = 0. \quad (77)$$

Noting (63) leads to

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\frac{2 \tan t}{x}. \quad (78)$$

Let

$$\rho = 2 \ln |x|. \quad (79)$$

Then the second equation in (78) becomes

$$\frac{dt}{d\rho} = -\tan t. \quad (80)$$

Thus, in the (t, ρ) -plan the null curves and light-cones are shown in Figure 1 in which x should be replaced by ρ .

We now study the geometric behavior of the t -slices.

For any fixed $t \in \mathbb{R}$, the induced metric of the t -slice reads

$$ds^2 = -\frac{\sin^2 t}{x^4} [dy^2 + (2 + \cos y)^{-2} dz^2]. \quad (81)$$

When $t = k\pi$ ($k \in \mathbb{N}$), the metric (81) becomes

$$ds^2 = 0.$$

This implies that the t -slice reduces to a point. On the other hand, in this case the metric (61) becomes

$$ds^2 = \frac{2}{(2 + \cos y)x^2} dt^2.$$

When $t \neq k\pi$ ($k \in \mathbb{N}$), (81) shows that the t -slice is a three-dimensional manifold with cone-like singularities at $x = \infty$ and $x = -\infty$, respectively. In particular, if we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) , then the induced metric (81) becomes

$$ds^2 = -\frac{\sin^2 t}{r^4} [d\theta^2 + (2 + \cos \theta)^{-2} d\varphi^2]. \quad (82)$$

In this case the t -slice is a three-dimensional cone-like manifold centered at $r = \infty$.

At the end of this subsection, we would like to emphasize that the space-time (61) possesses a physical singularity, i.e., $x = 0$ which is named as a quasi-black-hole in this paper.

4. Summary and discussion. In this paper we describe a new method to find exact solutions of the Einstein's field equations (1). Using our method, we can construct many interesting exact solutions, in particular, the time-periodic solutions of the vacuum Einstein's field equations. More precisely, we have constructed three kinds

of new time-periodic solutions of the vacuum Einstein's field equations: the regular time-periodic solution with vanishing Riemann curvature tensor, the regular time-periodic solution with finite Riemann curvature tensor and the time-periodic solution with physical singularities. We have also analyzed the singularities of these new time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times.

In particular, in the spherical coordinates (t, r, θ, φ) we construct a time-periodic space-time with essential singularities. This space-time possesses an interesting and important singularity which is named as a *quasi-black-hole*. This space-time is inhomogenous and not asymptotically flat and can perhaps be used to explain the phenomenon that our Universe exists anisotropy from the recent WMAP data (see [2]). We believe some applications of these new space-times in modern cosmology and general relativity can be expected.

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- [1] J. Bičák, *Selected solutions of Einstein's field equations: their role in general relativity and astrophysics*, in Einstein's field equations and their physical implications, Lecture Notes in Phys. **540**, Springer, Berlin, 2000, 1-126.
 - [2] G. Hinshaw, et al., arXiv:astro-ph/0603451.
 - [3] D.-X. Kong and K.-F. Liu, Time-periodic solutions of the Einstein's field equations, arXiv:0805.1100v2.
 - [4] D.-X. Kong and K.-F. Liu, Time-periodic cosmology, to appear.
 - [5] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations* (second edition), Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2003.
 - [6] R. M. Wald, *General relativity*, the University of Chicago Press, Chicago and London, 1984.